XXIV. *On the Values of the Integral* $\int_0^1 Q_n Q_{n'} d\mu$, Q_n, $Q_{n'}$ *being* LAPLACE'S *Coefficients of the Orders* n, n', *with an application to the Theory of Radiation.*

By the Hon. J. W. STRUTT, *Fellow of Trinity College, Cambridge. Communicated by* W. SPOTTISWOODE, *F.R.S.*

Received May 17,—Read June 16, 1870.

IN the course of an investigation concerning the potential function which is subject to conditions at the surface of a sphere which vary discontinuously in passing from one hemisphere to the other, it became necessary to know the values of the integral

$$\int_0^1 Q_n Q_{n'} d\mu,$$

Q_n, $Q_{n'}$ being LAPLACE'S coefficients of the orders n, n' respectively. The expression for Q_n in terms of μ is

$$Q_n=\frac{1.3.5\ldots.(2n-1)}{1.2.3..n}\left\{\mu^n-\frac{n(n-1)}{2.(2n-1)}\mu^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}\mu^{n-4}-\ldots\right\}*;$$

but the multiplication of two such series together and subsequent integration with respect to μ would be very laborious even for moderate values of n and n'.

By the following method the values of the integrals in question may be obtained without much trouble. According to the definition of the functions Q,

$$\frac{1}{\sqrt{1+e^2-2e\mu}}=1+Q_1e+Q_2e^2+\ldots+Q_ne^n+\ldots$$

so that

$$\int_0^1\frac{d\mu}{\sqrt{1+e^2-2e\mu}\,\sqrt{1+e'^2-2e'\mu}}=\sum_{n=0}^{n=\infty}\sum_{n'=0}^{n'=\infty}\int_0^1 Q_nQ_{n'}d\mu\,.\,e^ne'^{n'},$$

which shows that $\int_0^1 Q_nQ_{n'}d\mu$ is the coefficient of $e^ne'^{n'}$ in the expansion of the integral on the left in powers of e and e'.

On effecting the integration and reducing, we obtain as the quantity to be expanded,

$$\frac{1}{\sqrt{ee'}}\log\frac{(1+\sqrt{ee'})(\sqrt{e}-\sqrt{e'})}{\sqrt{e}\sqrt{1+e'^2}-\sqrt{e'}\sqrt{1+e^2}}=\frac{1}{\sqrt{ee'}}\log(1+\sqrt{ee'})-\frac{1}{\sqrt{ee'}}\left\{\log\sqrt{1+e^2}+\log\frac{1-\sqrt{\frac{1+e'^2}{1+e^2}}\sqrt{\frac{e}{e'}}}{1-\sqrt{\frac{e}{e'}}}\right\}$$

$$=\frac{1}{\sqrt{ee'}}\left\{(ee')^{\frac{1}{2}}-\frac{ee'}{2}+\frac{(ee')^{\frac{3}{2}}}{3}-\frac{(ee')^2}{4}+\ldots\right\}$$

* THOMSON and TAIT'S Natural Philosophy, p. 624.

$$+\frac{1}{\sqrt{ee'}}\left[\begin{array}{l}\left\{\left(\frac{1+e'^2}{1+e^2}\right)^{\frac{1}{2}}-1\right\}\left(\frac{e}{e'}\right)^{\frac{1}{2}}+\frac{1}{2}\left\{\frac{1+e'^2}{1+e^2}-1\right\}\frac{e}{e'}\\ +\frac{1}{3}\left\{\left(\frac{1+e'^2}{1+e^2}\right)^{\frac{3}{2}}-1\right\}\left(\frac{e}{e'}\right)^{\frac{3}{2}}+\frac{1}{4}\left\{\left(\frac{1+e'^2}{1+e^2}\right)^{2}-1\right\}\left(\frac{e}{e'}\right)^{2}\\ +\ \ldots\ldots\ldots\ +\ \ldots\ldots\ldots\\ -\frac{1}{2}\log(1+e^2).\end{array}\right]$$

Since we know *à priori* that the expansion will only contain whole positive powers of e, we may leave out all the terms in $e^{\frac{1}{2}}e^{\frac{3}{2}}\ldots\ldots$ (no negative powers of e occur)*. We thus obtain

$$\begin{array}{l}1+\frac{ee'}{3}+\frac{(ee')^2}{5}+\frac{(ee')^3}{7}+\frac{(ee')^4}{9}+\ldots\ldots\\ +\frac{1}{e'}\left\{\left(\frac{1+e'^2}{1+e^2}\right)^{\frac{1}{2}}-1\right\}+\frac{1}{3}\frac{e}{e'^2}\left\{\left(\frac{1+e'^2}{1+e^2}\right)^{\frac{3}{2}}-1\right\}\\ +\frac{1}{5}\frac{e^2}{e'^3}\left\{\left(\frac{1+e'^2}{1+e^2}\right)^{\frac{5}{2}}-1\right\}+\frac{1}{7}\frac{e^3}{e'^4}\left\{\left(\frac{1+e'^2}{1+e^2}\right)^{\frac{7}{2}}-1\right\}\\ +\frac{1}{9}\frac{e^4}{e'^5}\left\{\left(\frac{1+e'^2}{1+e^2}\right)^{\frac{9}{2}}-1\right\}+\frac{1}{11}\frac{e^5}{e'^6}\left\{\left(\frac{1+e'^2}{1+e^2}\right)^{\frac{11}{2}}-1\right\}\\ +\ldots\ldots\qquad\qquad +\ldots\ldots\end{array}$$

In this all terms containing negative powers of e' may be thrown away, as they must finally disappear even if retained. The terms on the left *after the first line* contain only even powers of e, and those on the right only odd. It appears, too, that with the even powers of e go the odd of e', and conversely. Hence if n, n' be both even or both odd there is no part of the coefficient of $e^n e'^{n'}$ to be found after the first line, and none in the first line unless $n=n'$. Thus

$$\int_0^1 Q_n Q_{n'}\,d\mu=0$$

if n, n' are both odd or both even, unless they are the same, in which case

$$\int_0^1 (Q_n)^2\,d\mu=\frac{1}{2n+1}.$$

* Professor CAYLEY has remarked that the finite expression itself may be modified so as to get rid of these terms, and then becomes

$$\frac{1}{2\sqrt{ee'}}\log\frac{1+\sqrt{ee'}}{1-\sqrt{ee'}}+\frac{1}{2\sqrt{ee'}}\left\{\log\frac{1+\sqrt{\frac{e(1+e'^2)}{e'(1+e^2)}}}{1-\sqrt{\frac{e(1+e'^2)}{e'(1+e^2)}}}-\log\frac{1+\sqrt{\frac{e}{e'}}}{1-\sqrt{\frac{e}{e'}}}\right\}.$$

For so far as the terms containing fractional powers of x are concerned,

$$\log(1+\sqrt{x})\text{ and }\tfrac{1}{2}\log\frac{1+\sqrt{x}}{1-\sqrt{x}}$$

are identical.—(Nov. 1870.)

These results* are immediate consequences of what is known with respect to the values of the integrals

$$\int_{-1}^{+1} \mathrm{Q}_n \mathrm{Q}_{n'}\, d\mu,$$

in which the integration extends over the *whole* sphere; for if n, n' are both odd or both even, $\mathrm{Q}_n\mathrm{Q}_{n'}$ is an even function of μ, and so

$$\int_{-1}^{0} \mathrm{Q}_n \mathrm{Q}_{n'}\, d\mu = \int_{0}^{1} \mathrm{Q}_n \mathrm{Q}_{n'}\, d\mu.$$

The peculiar character of the integrals over the hemisphere only shows itself when one of the quantities n, n' is even and the other odd.

The coefficient of e^0 in the expansion is $1+\dfrac{(1+e'^2)^{\frac{1}{2}}-1}{e'}$;

$$\text{coefficient of } e^2 = \frac{e'^2}{5} - \frac{1}{2}\frac{(1+e'^2)^{\frac{1}{2}}}{e'} + \frac{1}{5}\frac{(1+e'^2)^{\frac{5}{2}}-1}{e'^3};$$

$$\text{coefficient of } e^4 = \frac{e'^4}{9} + \frac{1.3}{2^2\underline{|2}}\frac{(1+e'^2)^{\frac{1}{2}}}{e'} - \frac{5}{2}\cdot\frac{1}{5}\frac{(1+e'^2)^{\frac{5}{2}}}{e'^3} + \frac{1}{9}\frac{(1+e'^2)^{\frac{9}{2}}-1}{e'^5};$$

$$\text{coefficient of } e^6 = \frac{e'^6}{13} - \frac{1.3.5}{2^3\underline{|3}}\frac{(1+e'^2)^{\frac{1}{2}}}{e'} + \frac{5.7}{2^2\underline{|2}}\cdot\frac{1}{5}\frac{(1+e'^2)^{\frac{5}{2}}}{e'^3} - \frac{9}{2}\cdot\frac{1}{9}\frac{(1+e'^2)^{\frac{9}{2}}}{e'^5} + \frac{1}{13}\frac{(1+e'^2)^{\frac{13}{2}}-1}{e'^7};$$

$$\text{coefficient of } e^8 = \frac{e'^8}{17} + \frac{1.3.5.7}{2^4\underline{|4}}\frac{(1+e'^2)^{\frac{1}{2}}}{e'} - \frac{5.7.9}{2^3\underline{|3}}\cdot\frac{1}{5}\frac{(1+e'^2)^{\frac{5}{2}}}{e'^3}$$
$$+ \frac{9.11}{2^2\underline{|2}}\cdot\frac{1}{9}\frac{(1+e'^2)^{\frac{9}{2}}}{e'^5} - \frac{13}{2}\cdot\frac{1}{13}\frac{(1+e'^2)^{\frac{13}{2}}}{e'^7} + \frac{1}{17}\frac{(1+e'^2)^{\frac{17}{2}}-1}{e'^9}.$$

The law of formation of these series is obvious, and the coefficient of e^{2n} could, if necesssary, be written down.

From the symmetry of the original expression in e and e' we know that the coefficient of $e^n e'^{n'}$ must be the same as that of $e^{n'}e'^n$; so that, in order to obtain all the integrals required, it is not absolutely necessary to consider the coefficients of odd powers of e. Nevertheless, in the calculation for instance of $\int_0^1 \mathrm{Q}_1\mathrm{Q}_{10}\, d\mu$, it would be much easier to obtain it as the coefficient of e'^{10} in the series which multiplies e, than as the coefficient of e' in the series which multiplies e^{10}.

The coefficient of e

$$= \frac{e'}{3} + \frac{1}{3}\frac{(1+e'^2)^{\frac{3}{2}}-1}{e'^2};$$

coefficient of e^3

$$= \frac{e'^3}{7} - \frac{3}{2}\cdot\frac{1}{3}\frac{(1+e'^2)^{\frac{3}{2}}}{e'^2} + \frac{1}{7}\frac{(1+e'^2)^{\frac{7}{2}}-1}{e'^4};$$

* They would, of course, be more simply obtained by taking the integration in the first instance from $\mu=-1$ to $\mu=+1$.—(Nov. 1870.)

coefficient of e^5

$$=\frac{e'^5}{11}+\frac{3.5}{2^2\lfloor 2}\cdot\frac{1}{3}\frac{(1+e'^2)^{\frac{3}{2}}}{e'^2}-\frac{7}{2}\cdot\frac{1}{7}\frac{(1+e'^2)^{\frac{7}{2}}}{e'^4}+\frac{1}{11}\frac{(1+e'^2)^{\frac{11}{2}}-1}{e'^6};$$

coefficient of e^7

$$=\frac{e'^7}{15}-\frac{3.5.7}{2^3\lfloor 3}\cdot\frac{1}{3}\frac{(1+e'^2)^{\frac{3}{2}}}{e'^2}+\frac{7.9}{2^2\lfloor 2}\cdot\frac{1}{7}\frac{(1+e'^2)^{\frac{7}{2}}}{e'^4}$$

$$-\frac{11}{2}\cdot\frac{1}{11}\frac{(1+e'^2)^{\frac{11}{2}}}{e'^6}+\frac{1}{15}\frac{(1+e'^2)^{\frac{15}{2}}-1}{e'^8};$$

and so on. From these series the coefficients of $e^n e'^{n'}$ for moderate values of n and n' may be calculated with facility.

It is desirable to know the limit of the integral $\int_0^1 Q_n Q_{n'} d\mu$ when n becomes very large, n' remaining finite. A distinction is necessary according as it is the even or the odd suffix which is supposed to increase without limit.

The whole coefficient of e^{2n} is a sum of terms of the form $\frac{(1+e'^2)^{\frac{4m+1}{2}}}{e'^{2m+1}}$, where m is zero, or any positive integer, each term multiplied by a numerical factor, which may be regarded as a function of n and m. The general term in the expansion of $\frac{(1+e'^2)^{\frac{4m+1}{2}}}{e'^{2m+1}}$ in powers of e' is

$$e'^{2r-2m-1}\frac{(4m+1)(4m-1)\ldots\ldots3.1.1.3\ldots(2r-4m-3)}{2^r\lfloor r},$$

irrespective of sign.

If we put $2r-2m-1=2n'-1$, it becomes

$$e'^{2n'-1}\frac{(4m+1)(4m-1)\ldots3.1.1.3\ldots(2n'-2m-3)}{2.4.6\ldots(2n'+2m)}.$$

The coefficient of $e^{2n}e'^{2n'-1}$ is thus a series of terms of the form

$$\frac{(4m+1)\ldots\ldots1.1\ldots.(2n'-2m-3)}{2.4.6\ldots(2n'+2m)},$$

each term multiplied by a factor depending on n and m but *independent of* n'. The question is which term has the predominance when n' increases without limit? It appears that it is the one corresponding to $m=0$; for the ratio of this to the general term is

$$\frac{1.1.3\ldots.(2n'-3)}{(4m+1)\ldots\ldots1.1\ldots.(2n'-2m-3)}\cdot\frac{2.4\ldots\ldots(2n'+2m)}{2.4\ldots2n'}$$

$$=\frac{(2n'-2m-1)\ldots\ldots(2n'-3)\times(2n'+2)\ldots\ldots(2n'+2m)}{(4m+1)(4m-1)\ldots1},$$

a fraction which increases without limit with n'.

The value of $\int_0^1 Q_{2n} Q_{2n'-1}\,d\mu$, when n' is indefinitely great, is therefore identical with the coefficient of $e'^{2n'-1}$ in the expansion of

$$\frac{1.3.5\ldots..(2n-1)}{2.4.6\ldots.2n}\frac{(1+e'^2)^{\frac{1}{2}}}{e'}.$$

Now

$$\frac{(1+e'^2)^{\frac{1}{2}}}{e'}=\frac{1}{e'}+\frac{1}{2}e'+\ldots.\pm\frac{1.1.3\ldots.(2n'-3)}{2.4.6\ldots.2n'}e'^{2n'-1};$$

and by a known theorem, when n' is indefinitely great,

$$\frac{1.1.3\ldots..(2n'-3)}{2.4.6\ldots2n'}=\frac{1}{2n'-1}\,\frac{1}{\pi^{\frac{1}{2}}n'^{\frac{1}{2}}}.$$

Finally, therefore, $\int_0^1 Q_{2n} Q_{2n'-1}\,d\mu$, when n' increases without limit, takes ultimately the form $\pm\frac{1.3.5\ldots.(2n-1)}{2.4.6\ldots2n}\frac{1}{2\pi^{\frac{1}{2}}n'^{\frac{3}{2}}}$.

If n be very great (though infinitely small, perhaps, compared to n') this becomes $\pm\frac{1}{2\pi n^{\frac{1}{2}}n'^{\frac{3}{2}}}$. In a similar manner it may be proved that the limit of $\int_0^1 Q_{2n-1} Q_{2n'}\,d\mu$, when n' increases indefinitely, is

$$\pm\frac{3.5.7\ldots(2n-1)}{2.4.6\ldots.(2n-2)}\cdot\frac{1}{4\pi^{\frac{1}{2}}n'^{\frac{5}{2}}}.$$

If, now, n increases without limit, we obtain

$$\frac{n^{\frac{1}{2}}}{2\pi n'^{\frac{5}{2}}}.$$

There is no inconsistency in the non-agreement of the values found when n and n' are indefinitely great, for the limiting circumstances contemplated in the two cases are in reality quite different. It may be convenient for the sake of comparison to repeat here the equation

$$\int_0^1 (Q_n)^2 d\mu=\frac{1}{2n+1},$$

which is true whether n be great or small.

The annexed Table contains the exact numerical values of the integrals for which neither suffix is greater than eleven. If we fix our attention on a given value of n (say 6), while n' varies, we see that the integrals, or, rather, those of them which do not vanish, begin by being alternately opposite in sign, and increase in value up to $n'=n-1$; that when $n'-n$ a change of sign is missed, and that for greater values of n' the regular alternation of sign is reestablished in conjunction with a steady diminution in numerical value.

This is in accordance with what might have been expected from the general character of the functions Q. They have their maximum (arithmetical as well as algebraical) value, namely unity, when $\mu=1$, an even function, Q_{2n}, vanishing n times, and an odd function, Q_{2n+1}, vanishing $n+1$ times for values of μ ranging from 0 to 1 inclusive.

$\int_0^1 Q_n Q_{n'} d\mu$.	Q_0.	Q_1.	Q_2.	Q_3.	Q_4.	Q_5.	Q_6.	Q_7.	Q_8.	Q_9.	Q_{10}.	Q_{11}.	Q_{12}.	Q_{13}.	Q_{14}.	Q_{15}.
Q_0	1	$\frac{1}{2}$	0	$-\frac{1}{8}$	0	$\frac{1}{16}$	0	$-\frac{5}{128}$	0	$\frac{7}{256}$	0	$-\frac{21}{1024}$	0	$\frac{33}{2048}$	0	$-\frac{13.33}{2^{15}}$
Q_1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{8}$	0	$-\frac{1}{48}$	0	$\frac{1}{128}$	0	$-\frac{1}{2^8}$	0	$\frac{7}{3.2^{10}}$	0	$-\frac{3}{2048}$	0	$\frac{33}{2^{15}}$	0
Q_2	0	$\frac{1}{8}$	$\frac{1}{5}$	$\frac{1}{8}$	0	$-\frac{5}{128}$	0	$\frac{7}{5.2^6}$	0	$-\frac{15}{2^{10}}$	0	$\frac{11}{2^{10}}$	0	$-\frac{273}{2^{15}}$	0	
Q_3	$-\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{1}{7}$	$\frac{9}{128}$	0	$-\frac{1}{2^6}$	0	$\frac{7}{2^{10}}$	0	$-\frac{54}{7.2^{11}}$	0	......	0	......	0
Q_4	0	$-\frac{1}{48}$	0	$\frac{9}{128}$	$\frac{1}{9}$	$\frac{9}{128}$	0	$-\frac{70}{3.2^{10}}$	0	$\frac{27}{2^{11}}$	0	$-\frac{297}{2^{15}}$	0	$\frac{1001}{2^{14}.9}$	0	
Q_5	$\frac{1}{16}$	0	$-\frac{5}{128}$	0	$\frac{9}{128}$	$\frac{1}{11}$	$\frac{25}{2^9}$	0	$-\frac{25}{2^{11}}$	0	$\frac{189}{2^{15}}$	0	......	0	......	0
Q_6	0	$\frac{1}{128}$	0	$-\frac{1}{2^5}$	0	$\frac{25}{2^9}$	$\frac{1}{13}$	$\frac{25}{2^9}$	0	$-\frac{525}{2^{15}}$	0	$\frac{77}{2^{13}}$	0	......	0	
Q_7	$-\frac{5}{128}$	0	$\frac{7}{5.2^6}$	0	$-\frac{70}{3.2^{10}}$	0	$\frac{25}{2^9}$	$\frac{1}{15}$	$\frac{1225}{2^{15}}$	0	$-\frac{383}{2^{11}.3}$	0	......	0	......	0
Q_8	0	$-\frac{1}{2^8}$	0	$\frac{7}{2^{10}}$	0	$-\frac{25}{2^{11}}$	0	$\frac{1225}{2^{15}}$	$\frac{1}{17}$	$\frac{1225}{2^{15}}$	0	$-\frac{1617}{2^{17}}$	0	......	0	
Q_9	$\frac{7}{256}$	0	$-\frac{15}{2^{10}}$	0	$\frac{27}{2^{11}}$	0	$\frac{525}{2^{15}}$	0	$\frac{1225}{2^{15}}$	$\frac{1}{19}$	$\frac{3969}{2^{17}}$	0	......	0	......	0
Q_{10}	0	$\frac{7}{3.2^{10}}$	0	$-\frac{54}{7.2^{11}}$	0	$\frac{189}{2^{15}}$	0	$-\frac{383}{2^{11}.3}$	0	$\frac{3969}{2^{17}}$	$\frac{1}{21}$	$\frac{3969}{2^{17}}$	0	......	0	
Q_{11}	$-\frac{21}{1024}$	0	$\frac{11}{2^{10}}$	0	$-\frac{297}{2^{15}}$	0	$\frac{77}{2^{13}}$	0	$-\frac{1617}{2^{17}}$	0	$\frac{3969}{2^{17}}$	$\frac{1}{23}$	......	0	......	0

When, therefore, one of the quantities n, n' is large and the other not nearly equal to it, $Q_nQ_{n'}$ is affected with a sign rapidly alternating, and consequently the value of the $\int_0^1 Q_nQ_{n'}d\mu$ is comparatively very small. But if n and n' are nearly equal, the functions Q_n, $Q_{n'}$, in spite of the rapid alternation, keep together as it were in sign for a considerable fraction of the range of integration, and so the value of the integral is largely increased.

Again, for all cases included in the Table it will be found that

$$\int_0^1 Q_{2n}Q_{2n-1}d\mu=\int_0^1 Q_{2n}Q_{2n+1}d\mu,$$

a relation which is evidently general, although not very easily proved to be so. After a good deal of trouble I arrived at the following demonstration:—

If in the expression

$$\frac{1}{\sqrt{ee'}}\log\frac{(1+\sqrt{ee'})\left(1-\sqrt{\frac{e}{e'}}\right)}{\sqrt{1+e^2}-\sqrt{\frac{e}{e'}}\sqrt{1+e'^2}},$$

whose expansion gives the integrals under consideration, we put

$$ee'=x,\quad \frac{e}{e'}=y,$$

we obtain

$$\frac{1}{\sqrt{x}}\log\frac{(1+\sqrt{x})(1-\sqrt{y})}{\sqrt{1+xy}-\sqrt{y}\sqrt{1+\frac{x}{y}}}.$$

In consequence of the symmetry of this in respect to y and $\frac{1}{y}$, it may be expanded in a series of positive and negative powers of $\sqrt{y}$ of the form

$$A_0+A_1\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right)+A_2\left(y+\frac{1}{y}\right)+A_3(y^{\frac{3}{2}}+y^{-\frac{3}{2}})+\ldots,$$

A_0, A_1, ... being functions of x.

The terms that we are engaged in examining are those in $\sqrt{y}$ or $\frac{1}{\sqrt{y}}$, so that the question reduces itself to the determination of A_1 as a function of x, or at least an examination of its nature.

Now A_1 is the term independent of y after differentiation of the series with respect to $\sqrt{y}$. Hence

$$A_1=\text{term independent of } y \text{ in } \frac{d}{d\sqrt{y}}\cdot\frac{1}{\sqrt{x}}\log\frac{(1+\sqrt{x})(1-\sqrt{y})}{\sqrt{1+xy}-\sqrt{y}\sqrt{1+\frac{x}{y}}}.$$

On differentiation and reduction we arrive at the expression

$$-\frac{1}{\sqrt{x}(1-y)}+\frac{1+x}{\sqrt{x}}(1-y)^{-1}\left\{1+x^2+x\left(y+\frac{1}{y}\right)\right\}^{-\frac{1}{2}},$$

from which the part independent of y has to be selected.

From the first term we have simply $\frac{-1}{\sqrt{x}}$. As for the second,

$$\left\{1+x^2+x\left(y+\frac{1}{y}\right)\right\}^{-\frac{1}{2}}$$

$$=(1+x^2)^{-\frac{1}{2}}\left\{1-\frac{1}{2}\frac{x}{1+x^2}\left(y+\frac{1}{y}\right)+\frac{1.3}{2^2\underline{|2}}\frac{x^2}{(1+x^2)^2}\left(y+\frac{1}{y}\right)^2\right.$$

$$\left.-\frac{1.3.5}{2^3\underline{|3}}\frac{x^3}{(1+x^2)^3}\left(y+\frac{1}{y}\right)^3+\dots\right\}.$$

The term in $\left(y+\frac{1}{y}\right)(1+y+y^2+y^3+\dots)$ independent of y is 1.

The term in $\left(y+\frac{1}{y}\right)^3(1+y+y^2+\dots)$ is $1+3$, and generally, *if n be odd*, the part independent of y in

$$\left(y+\frac{1}{y}\right)^n(1-y)^{-1} \text{ is } \frac{1}{2}(1+1)^n=\frac{1}{2}\cdot 2^n.$$

Thus the term in $\left\{1+x^2+x\left(y+\frac{1}{y}\right)\right\}^{-\frac{1}{2}}(1-y)^{-1}$ independent of y

$$=(1+x^2)^{-\frac{1}{2}}\left\{\begin{array}{l}-\frac{1}{2}\,\frac{x}{1+x^2}\,\frac{1}{2}\cdot 2-\frac{1.3.5}{2^3\underline{|3}}\,\frac{x^3}{(1+x^2)^3}\,\frac{1}{2}\cdot 2^2 \\ -\dots\dots+\text{ an } \textit{even} \text{ function of } x\end{array}\right\}$$

$$=\frac{(1+x^2)^{-\frac{1}{2}}}{2}\left\{\begin{array}{l}1-\frac{1}{2}\cdot\frac{2x}{1+x^2}+\frac{1.3}{2^2\underline{|2}}\,\frac{(2x)^2}{(1+x^2)^2}-\frac{1.3.5}{2^3\underline{|3}}\,\frac{(2x)^3}{(1+x^2)^3} \\ +\dots\dots+\text{ an even function of } x\end{array}\right\}$$

$$=\frac{(1+x^2)^{-\frac{1}{2}}}{2}\left[\left\{1+\frac{2x}{1+x^2}\right\}^{-\frac{1}{2}}+\text{ even function of } x\right]$$

$$=\frac{1}{2}\,\frac{1}{1+x}+\text{ an even function of } x.$$

Finally, therefore,

$$A_1=-\frac{1}{2\sqrt{x}}+\frac{1+x}{\sqrt{x}}\left\{a_0+a_2x^2+a_4x^4+\dots\right\},$$

where a_0, $a_2\dots$ are unknown coefficients, of which we will only determine a_0 by a reference to the expression from which A_1 was obtained, which shows that

$$-\tfrac{1}{2}+a_0=0,$$

so that

$$A_1=\tfrac{1}{2}\sqrt{x}+a_2x^{\frac{3}{2}}+a_2x^{\frac{5}{2}}+a_4x^{\frac{7}{2}}+a_4x^{\frac{9}{2}}+\dots.$$

Multiplying by $\sqrt{y}$ and replacing e and e',

$$A\sqrt{y}=\tfrac{1}{2}e+a_2e^2e'+a_2e^3e'^2+a_4e^4e'^3+a_4e^5e'^4+a_6e^6e'^5+a_6e^7e'^6+\dots.$$

These are the only terms in the expansion of $\frac{1}{\sqrt{ee'}}\log\frac{(1+\sqrt{ee'})\left(1-\sqrt{\frac{e}{e'}}\right)}{\sqrt{1+e^2}-\sqrt{\frac{e}{e'}}\sqrt{1+e'^2}}$ in which the index of e is *one* higher than that of e'.

Having regard, now, to the symmetry in e and e', we see that generally

$$\int_0^1 Q_{2n}Q_{2n-1}d\mu=\int_0^1 Q_{2n}Q_{2n+1}d\mu.$$

As an application of some of the results of this investigation I will take the following physical problem. A spherical ball of uniform material is exposed to the radiation from infinitely distant surrounding objects. It is required to find the *stationary* condition. For the sake of simplicity, the surface of the sphere will be supposed to be perfectly black, that is, to absorb all the radiant heat that falls upon it, and NEWTON'S law of cooling will be employed, at least provisionally.

If V denote the temperature, it is to be determined by the equations

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)V=0, \quad \ldots\ldots\ldots\ldots \quad \text{(A)}$$

$$k\frac{dV}{dr}=F(E)-hV, \quad \ldots\ldots\ldots\ldots \quad \text{(B)}$$

where F(E) is a function of the position of the point E on the surface, and denotes the heat received per unit area at that point, k is the conductivity, and h the coefficient of radiation. Equation (A) is to be satisfied throughout the interior and (B) over the surface of the sphere.

If V be expanded in LAPLACE'S series,

$$V=S_0+S_1\frac{r}{a}+S_2\frac{r}{a^2}+\ldots; \quad \frac{dV}{dr=a}=\frac{1}{a}(S_1+2S_2+3S_3+\ldots);$$

and if

$$F=F_0+F_1+F_2+\ldots.$$

be the expansion of F in a similar series of surface harmonics, we obtain, on substituting in (B) and equating to zero the terms of any order,

$$S_0=\frac{F_0}{h},$$

$$S_1=\frac{F_1}{h+\frac{k}{a}},$$

$$S_2=\frac{F_2}{h+\frac{2k}{a}},$$

$$\ldots\ldots$$

$$S_n=\frac{F_n}{h+\frac{nk}{a}}. \quad \ldots\ldots\ldots\ldots \quad \text{(C)}$$

The mean temperature S_0 is seen to be independent of the conductivity and of the size of the sphere.

The case where the heat which falls on the sphere proceeds from a single radiant-point is not only important in itself, but may be made the foundation of the general solution in virtue of the principle of superposition. Taking the axis in the direction of the radiant-point, we have

$$F(E)=\mu$$

over the positive hemisphere, that is, from $\theta=0$ to $\theta=\frac{\pi}{2}$,

while over the negative hemisphere $F(E)=0$.

It is required to expand F in a series of spherical harmonics.

Let $F=\frac{1}{2}\mu+f$, then f is a function of μ, which is equal to $\frac{1}{2}\mu$ over the positive hemisphere and to $-\frac{1}{2}\mu$ over the negative. The problem therefore reduces itself to the expression of $\frac{1}{2}\mu$ over the positive hemisphere in a series of functions Q *of even order*. The same series will then give $-\frac{1}{2}\mu$ over the negative hemisphere.

Assume

$$\tfrac{1}{2}\mu=A_0+A_2Q_2+A_4Q_4+\ldots\ldots$$

Multiplying by Q_{2n}, and integrating with respect to μ from $\mu=0$ to $\mu=1$,

$$\tfrac{1}{2}\int_0^1 Q_1Q_{2n}d\mu=A_{2n}\int_0^1(Q_{2n})^2d\mu,$$

all the other terms on the right vanishing.

Now $\int_0^1(Q_{2n})^2d\mu=\frac{1}{4n+1}$.

$$\int_0^1 Q_1Q_{2n}d\mu=\text{coefficient of } e^{2n} \text{ in the expansion of}$$

$$\frac{1}{3}\,\frac{(1+e^2)^{\frac{3}{2}}-1}{e^2}$$

$$=-(-1)^n\,\frac{1.1.3.5\ldots.(2n-3)}{2.4.6\ldots.(2n+2)},$$

or

$$A_{2n}=-(-1)^n\,\frac{4n+1}{2}\,\frac{1.1.3\ldots.(2n-3)}{2.4.6\ldots.(2n+2)}.$$

Accordingly

$$F(E)=\tfrac{1}{4}+\tfrac{1}{2}Q_1+\tfrac{5}{16}Q_2-\tfrac{3}{32}Q_4+\ldots-(-1)^n\frac{4n+1}{2}\,\frac{1.1.3.5\ldots.(2n-3)}{2.4.6\ldots.(2n+2)}\,Q_{2n}+\ldots\ldots$$

When n is great the coefficient of Q_{2n} approximates to $\frac{1}{2\sqrt{\pi}.n^{\frac{3}{2}}}$.

This completes the solution for a sphere exposed to the radiation from an infinitely distant source of heat situated over the point $\mu=1$.

If its coordinates are μ', ϕ', it is only necessary to replace μ in Q_{2n} by

$$\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi').$$

Hence if H denote the intensity of the radiation which comes in *direction* μ', ϕ', the general value of S_n is

$$S_{2n}=-(-1)^n \frac{4n+1}{2\left(h+\frac{nk}{a}\right)} \frac{1.1.3.5\ldots(2n-3)}{2.4.6\ldots(2n+2)}$$

$$\times\iint HQ_{2n}(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi'))d\mu'd\phi',$$

the integration going *all* round the sphere.

Now $(4n+1)\iint HQ_{2n}\,d\mu'd\phi'$ is the same as $4\pi H_{2n}$, where H_{2n} is the harmonic element of H of order $2n$; so that

$$S_{2n}=-(-1)^n \frac{1.1.3\ldots(2n-3)}{2.4.6\ldots(2n+2)} \frac{2\pi H_{2n}}{h+\frac{2nk}{a}},$$

$$S_0=\frac{\pi}{h}H_0,$$

$$S_1=\frac{2\pi}{3\left(h+\frac{k}{a}\right)}H_1,$$

$$S_2=\frac{\pi}{4\left(h+\frac{2k}{a}\right)}H_2,$$

$$S_4=-\frac{\pi}{24\left(h+\frac{4k}{a}\right)}H_4.$$

It is remarkable that the odd terms in H (except H_1) are altogether without influence. The reason is simply that they do not affect the total heat falling on any point of the surface.

For this is expressed by

$$\int_0^1\int_0^{2\pi}\mu H_n d\mu d\phi,$$

the point considered being taken as pole of μ, which involves no loss of generality.

Now (Thomson and Tait, p. 149)

$$H_n=\sum_{s=0}^{s=n}(A_s\cos s\phi+B_s\sin s\phi)\Theta_n^s,$$

where Θ_n^s is a function of μ not containing ϕ.

When the integration with respect to ϕ is effected, all the terms will vanish except that whose coefficient is A_0. For this purpose, therefore, we may take

$$H_n=A_0\Theta_n^0 \text{ or } A_0'Q_n,$$

and we know that $\int_0^1\mu Q_n d\mu$ vanishes if n be odd and different from unity*.

* The proof given is sufficient for the object in view; but it may be well to notice that the essential thing is that the two surface harmonics which are multiplied together are either both odd or both even. A harmonic

The same thing is true for an ellipsoid or body of any figure which lies altogether on one side of every tangent plane, namely, that the terms of odd order in H (except one) are wholly without influence on it, and for the same reason.

We saw that in the case of a sphere the mean temperature was independent of the conductivity, and also of the size of the sphere; but this depends on NEWTON's law of cooling. A comparison, however, may be made which shall hold good whatever be the law of variation of radiation with temperature; for if the conducting-power of any uniform body (which need not be oval) be increased in the same proportion as its linear dimensions, a corresponding distribution of temperature will satisfy all the conditions. Conclusions of interest from a physical point of view may be deduced from the foregoing considerations, but I refrain from pursuing the subject at present, as the physical problem was only brought forward in illustration of the mathematical results developed in this paper.

---

of even order has identical values at opposite points of the sphere, and one of odd order has contrary values. The product of two harmonics which are either both even or both odd has therefore the same value when integrated over any portion of the sphere, or over what may be called the opposite portion, or as a particular case over two opposite hemispheres. The last two integrals are the halves of the integral over the whole sphere, which vanishes by a well-known property of these functions.